APPLIED MATHEMATICS FOR ENGINEERS
MIDTERM 1

Code : *MAT 210*
Acad.Year : *2018-19*
Semester : *Fall*
Date : *11.11.2018*
Time : *10:40*
Duration : *110 min*

Last Name :
Name : Solution
Student ID :
Signature :
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10 QUESTIONS ON 5 PAGES
TOTAL 100 POINTS

P1. (20)	P2. (20)	P3. (20)	P4. (20)	P5. (20)		Total. (100)

1. *(10pts)* Compute the approximate solution using Euler's method on the interval $\underline{[1,2]}$ with $h = \frac{1}{2}$

$$y' = 4xy - 2 \qquad \underline{y(1) = 1} \rightarrow \text{Forwards Euler with } \begin{cases} x_0 = 1 \\ y_0 = 1 \end{cases}$$

$x_0 = 1$

$y_0 = 1$

$$y_0' = 4x_0y_0 - 2$$
$$= 4\cdot1\cdot1 - 2 = 2$$

$$x_1 = x_0 + h$$
$$= 1 + \tfrac{1}{2} = \tfrac{3}{2}$$

$$y_1 = y_0 + h\cdot y_0'$$
$$= 1 + \tfrac{1}{2}\cdot 2 = 2$$

$$y_1' = 4x_1y_1 - 2$$
$$= 4\cdot\tfrac{3}{2}\cdot 2 - 2 = 10$$

$$x_2 = x_1 + h$$
$$= \tfrac{3}{2} + \tfrac{1}{2} = 2$$

$$y_2 = y_1 + h\cdot y_1'$$
$$= 2 + \tfrac{1}{2}\cdot 10 = 7$$

x	y
1	1
3/2	2
2	7

2. *(10pts)* $\mathbf{D}[y] = f(x)$ with $\begin{cases} y'(1) = 2 \\ y(2) = 1 \end{cases}$ has the following impulse responses with $h = \frac{1}{3}$ on $[1,2]$.

$$\boldsymbol{y}^{(1)} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \qquad \boldsymbol{y}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Write the solution to $\mathbf{D}[y] = \underline{9(x-1)}$ with the same boundary values, interval, and h.

$\hookrightarrow f(x) = 9(x-1)$

$\underline{x}$

$x_0 = 1$

$x_1 = 4/3$

$x_2 = 5/3$

$x_3 = 2$

$$\underline{f} = f(x_1)\cdot h\cdot \underline{\delta}^{(1)} + f(x_2)\cdot h\cdot \underline{\delta}^{(2)}$$
$$= 9(\tfrac{4}{3}-1)\cdot\tfrac{1}{3}\cdot\underline{\delta}^{(1)} + 9(\tfrac{5}{3}-1)\cdot\tfrac{1}{3}\cdot\underline{\delta}^{(2)}$$
$$= 1\cdot\underline{\delta}^{(1)} + 2\cdot\underline{\delta}^{(2)}$$
$$\Downarrow$$
$$\underline{y} = 1\cdot\underline{y}^{(1)} + 2\cdot\underline{y}^{(2)}$$
$$= \begin{bmatrix} -2 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 4 \end{bmatrix}}$$

3. *(2×10=20pts)* Discretize the following 2nd order differential equations. (Use any method.)

(A) $y'' = x$ with $\begin{cases} y(2) = 0 \\ y(6) = 0 \end{cases}$ on $[2,6]$ with $h = 1$. $\begin{cases} y_0 = 0 \\ y_{n+1} = 0 \end{cases}$ & $1/h = 1$

x	y
$x_0 = 2$	$y_0 = 0$
$x_1 = 3$	y_1
$x_2 = 4$	y_2
$x_3 = 5$	y_3
$x_4 = 6$	$y_4 = 0$

unknown

$$\overbrace{y'' = x}$$

$$1\cdot(y_2 - 2y_1 + \cancel{y_0}^{\,0}) = y_1'' = x_1 = 3$$

$$1\cdot(y_3 - 2y_2 + y_1) = y_2'' = x_2 = 4$$

$$1\cdot(\cancel{y_4}_{\,0} - 2y_3 + y_2) = \underbrace{y_3'' = x_3}_{} = 5$$

$$\boxed{\underbrace{\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}}_{d^2/dx^2} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}}$$

(B) $y'' + y' + y = \delta(x - \underset{x_3}{\underline{1/2}})$ with $\begin{cases} y(-1) = 2 \\ y'(1) = 0 \end{cases}$ on $[-1, 1]$ with $h = 1/2$. $\begin{cases} y_0 = 2 \\ y'_{n+1} = 0 \end{cases}$ & $1/h = 2$

$\hookrightarrow y_{n+1} = y_n$

x	y
$x_0 = -1$	$y_0 = 2$
$x_1 = -1/2$	y_1
$x_2 = 0$	y_2
$x_3 = 1/2$	y_3
$x_4 = 1$	$y_4 = y_3$

unknown

$$\overbrace{y'' + y' + y = \delta(x - x_3)}$$

$$4(y_2 - 2y_1 + \cancel{y_0}^{\,2}) + \frac{2}{2}(y_2 - \cancel{y_0}^{\,2}) + y_1 = y_1'' + y_1' + y_1 = \delta(x_1 - x_3) = 0$$

$$4(y_3 - 2y_2 + y_0) + \frac{2}{2}(y_3 - y_1) + y_2 = y_2'' + y_2' + y_2 = \delta(x_2 - x_3) = 0$$

$$4(\cancel{y_4}_{\,y_3} - 2y_3 + y_2) + \frac{2}{2}(\cancel{y_4}_{\,y_3} - y_2) + y_3 = \underbrace{y_3'' + y_3' + y_3 = \delta(x_3 - x_3)}_{} = 2$$

$$\boxed{\begin{gathered} \left(4\underbrace{\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}}_{d^2/dx^2} + \frac{2}{2}\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}}_{d/dx} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{1} \right) \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0-6 \\ 0 \\ 2 \end{bmatrix} \\ \Updownarrow \\ \begin{bmatrix} -7 & 5 & 0 \\ 3 & -7 & 5 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 2 \end{bmatrix} \end{gathered}}$$

4. *(2×5=10pts)* Divide. (If there is no solution, write "no solution".)

(A)
$$\begin{bmatrix} \textcircled{2} & 1 \\ 0 & \textcircled{3} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 0 \end{bmatrix}$$

No Pivot! → $\boxed{0 \quad 0 \quad 0} \Rightarrow 0=0$ okay.

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 0 \end{bmatrix} \Rightarrow 3y = 6 \hookrightarrow \underline{\underline{y=2}}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 0 \end{bmatrix} \Rightarrow 2x = 4-2 \hookrightarrow \underline{\underline{x=1}}$$

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

(B)
z is free!
$$\begin{bmatrix} \textcircled{2} & 1 & 2 \\ 0 & \textcircled{3} & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$\underline{\underline{z=0}}$

$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$3y=6 \rightarrow \underline{\underline{y=2}}$

$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$2x = 4-2 \rightarrow \underline{\underline{x=1}}$

$\underline{\underline{z = 1 \cdot z}}$

$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3y = -1 \rightarrow \underline{\underline{y = -1/3}}$

$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ -1/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x = -5/3 \rightarrow \underline{\underline{x = -5/6}}$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + z\begin{bmatrix} -5/6 \\ -1/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ -2 \\ 6 \end{bmatrix}}$$

5. *(5pts)* Find the best approximate solution to the equation
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$
$$\underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{c_1}\ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{c_2}\qquad \underbrace{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}}_{b}$$

<u>Normal Eqn</u>: $\begin{bmatrix} c_1 \cdot c_1 & c_1 \cdot c_2 \\ c_2 \cdot c_1 & c_2 \cdot c_2 \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} c_1 \cdot b \\ c_2 \cdot b \end{bmatrix}$

$$\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \frac{1}{6-4}\begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$$

(2×2 inverse) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

6. *(5pts)* Find the best fit function $f(t) = at + bt^2$ to the data

t	-1	0	1
$f(t)$	2	4	8

$$a\,t + b\,t^2 = f(t)$$

$$\begin{bmatrix} -1 & (-1)^2 \\ 0 & 0^2 \\ 1 & 1^2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

↑

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Normal Equation ⇝
$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\boxed{\hat{f}(t) = 3t + 5t^2}$$

7. *(10pts)* Use the given LU decomposition to divide.

$$\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}2&1&1\\0&3&1\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\6\\2\end{bmatrix}$$

Divide by $\underline{L}$:

$$\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}4\\6\\2\end{bmatrix}\begin{array}{l}\to a=4\\ \to b=6-4=2\\ \to c=2-6=-4\end{array}$$

$$\begin{bmatrix}2&1&1\\0&3&1\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\2\\-4\end{bmatrix}$$

Divide by $\underline{U}$:

$\to z=-4$

$\to 3y=2-(-4) \to y=2$

$\to 2x=4-(-2) \to x=3$

$$\boxed{\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\2\\-4\end{bmatrix}}$$

8. *(10pts)* Use the given QR decomposition to find the best approximate solution.

$$\begin{bmatrix}1&1&-1\\1&-1&1\\0&1&2\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\3\\2\end{bmatrix}$$

"Divide" by $\underline{Q}$:

$$\begin{bmatrix}1&1&-1\\1&-1&1\\0&1&2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}1\\3\\2\end{bmatrix}$$

$\Downarrow$ Normal Eqn

$$\begin{bmatrix}2&0&0\\0&3&0\\0&0&6\end{bmatrix}\begin{bmatrix}\hat{a}\\\hat{b}\\\hat{c}\end{bmatrix}=\begin{bmatrix}4\\0\\6\end{bmatrix}$$

$$\begin{bmatrix}\hat{a}\\\hat{b}\\\hat{c}\end{bmatrix}=\begin{bmatrix}4/2\\0/3\\6/6\end{bmatrix}=\begin{bmatrix}2\\0\\1\end{bmatrix}$$

$\Updownarrow$

$$\left(\begin{bmatrix}\hat{a}\\\hat{b}\\\hat{c}\end{bmatrix}=\begin{bmatrix}c_1\cdot b/c_1\cdot c_1\\c_2\cdot b/c_2\cdot c_2\\c_3\cdot b/c_3\cdot c_3\end{bmatrix}=\begin{bmatrix}4/2\\0/3\\6/6\end{bmatrix}\right)$$

Divide by $\underline{R}$:

$$\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}\hat{x}\\\hat{y}\\\hat{z}\end{bmatrix}=\begin{bmatrix}4/2\\0/3\\6/6\end{bmatrix}=\begin{bmatrix}2\\0\\1\end{bmatrix}$$

$$\boxed{\begin{bmatrix}\hat{x}\\\hat{y}\\\hat{z}\end{bmatrix}=\begin{bmatrix}2-0\\0-1\\1\end{bmatrix}=\begin{bmatrix}2\\-1\\1\end{bmatrix}}$$

9. *(10pts)* Compute the LU decomposition for $A = \begin{bmatrix} 1 & 3 & -3 \\ 1 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$$\begin{array}{cc} L & U \\ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} \textcircled{1} & 3 & -3 \\ 1 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix} \end{array} \begin{array}{l} \longleftarrow \text{Pivot Row} \\ \\ \\ \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1/1 & 1 & 0 \\ 1/1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -3 \\ 0 & -1 & 4 \\ 0 & -2 & 2 \end{bmatrix} \begin{array}{l} \\ \leftarrow r_2 - r_1 \\ \leftarrow r_3 - r_1 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -3 \\ 0 & \textcircled{-1} & 4 \\ 0 & -2 & 2 \end{bmatrix} \begin{array}{l} \\ \longleftarrow \text{Pivot Row} \\ \\ \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -2/-1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -3 \\ 0 & -1 & 4 \\ 0 & 0 & -6 \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow r_3 - 2r_2 \end{array}$$

$$\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -3 \\ 0 & -1 & 4 \\ 0 & 0 & -6 \end{bmatrix}}$$

10. *(10pts)* Compute the QR decomposition for $A = \begin{bmatrix} 1 & 3 & -3 \\ 1 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Column Operation Method

"Pivot Column"

$$A = \begin{bmatrix} 1 & 3 & -3 \\ 1 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$6/3 = 2 \qquad -3/3 = -1$$

$$c_2 - 2c_1 \qquad c_3 + 1 \cdot c_1$$

$$= \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 6/3 & -3/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

"Pivot Column"

$$= \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$-2/2 = \textcircled{-1}$$

$$c_3 + 1 \cdot c_2$$

$$\boxed{= \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}$$

G-S Method

$$\begin{bmatrix} 1 & 3 & -3 \\ 1 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix} \qquad Q = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{c_1} \qquad R = [1]$$

b

$$\begin{bmatrix} 1 & 3 & -3 \\ 1 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

b

Approx $Q\underline{x} = \underline{b}$

$$\hat{\underline{x}} = \frac{c_1 \cdot b}{c_1 \cdot c_1} = \frac{6}{3} = \textcircled{2}$$

$$\underline{e} = \underline{b} - Q\hat{\underline{x}} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} 2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$Q = \begin{bmatrix} \underbrace{\begin{matrix} 1 \\ 1 \\ 1 \end{matrix}}_{c_1} & \underbrace{\begin{matrix} 1 \\ 0 \\ -1 \end{matrix}}_{c_2} \end{bmatrix} \qquad R = \begin{bmatrix} 1 & \textcircled{2} \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -3 \\ 1 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

b

$$\hat{x} = \begin{bmatrix} c_1 \cdot b / c_1 \cdot c_1 \\ c_2 \cdot b / c_2 \cdot c_2 \end{bmatrix} = \begin{bmatrix} -3/3 \\ -2/2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\underline{e} = \underline{b} - Q\hat{\underline{x}} = \begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & -1 & -1 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$